

Transport in simple networks described by integrable discrete nonlinear Schrödinger equation

K. Nakamura^(1,4), Z.A. Sobirov⁽²⁾, D.U. Matrasulov⁽²⁾, S. Sawada⁽³⁾

⁽¹⁾*Faculty of Physics, National University of Uzbekistan, Vuzgorodok, Tashkent 100174, Uzbekistan*

⁽²⁾*Turin Polytechnic University in Tashkent, 17 Niyazov Str., 100093, Tashkent, Uzbekistan*

⁽³⁾*Department of Physics, Kwansei Gakuin University, Sanda 669-1337, Japan*

⁽⁴⁾*Department of Applied Physics, Osaka City University, Osaka 558-8585, Japan*

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We elucidate the case in which the Ablowitz-Ladik (AL) type discrete nonlinear Schrödinger equation (NLSE) on simple networks (e.g., star graphs and tree graphs) becomes completely integrable just as in the case of a simple 1-dimensional (1-d) discrete chain. The strength of cubic nonlinearity is different from bond to bond, and networks are assumed to have at least two semi-infinite bonds with one of them working as an incoming bond. The present work is a nontrivial extension of our preceding one (Sobirov *et al.*, Phys. Rev. E **81**, 066602 (2010)) on the continuum NLSE to the discrete case. We find: (1) the solution on each bond is a part of the universal (bond-independent) AL soliton solution on the 1-d discrete chain, but is multiplied by the inverse of square root of bond-dependent nonlinearity; (2) nonlinearities at individual bonds around each vertex must satisfy a sum rule; (3) under findings (1) and (2), there exist an infinite number of constants of motion. As a practical issue, with use of AL soliton injected through the incoming bond, we obtain transmission probabilities inversely proportional to the strength of nonlinearity on the outgoing bonds.

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I. INTRODUCTION

We shall investigate transport in networks with vertices and bonds which received a growing attention recently. The networks of practical importance are those of nonlinear waveguides and optical fibers [1], double helix of DNA [2], Josephson junction arrays with Bose-Einstein Condensates [3], vein networks in leaves [4, 5], etc.

Major theoretical concern so far, however, is limited to solving stationary states of the linear Schrödinger equation, and to obtaining the energy spectra in closed networks and transmission probabilities for open networks with semi-infinite leads [6–11]. Only a few studies treat the nonlinear Schrödinger equation on simple networks, which are still limited to the analysis of its stationary state [12, 13].

With introduction of the nonlinearity to the time-dependent Schrödinger equation, the network becomes to provide a nice playground where one can see interesting soliton propagations and nonlinear dynamics through the network [14–17], namely through an assembly of continuum line segments connected at vertices. Although there exist important analytical studies on the semi-infinite and finite chains [18–21], we find little exact analytical treatment of soliton propagation through net-

works within a framework of nonlinear Schrödinger equation (NLSE) [22, 23]. The subject is difficult due to the presence of vertices where the underlying chain should bifurcate or multi-furcate in general.

Recently, with a suitable boundary condition at each vertex we developed an exact analytical treatment of soliton propagation through networks within a framework of NLSE [25]. Under an appropriate relationship among values of nonlinearity at individual bonds, we found nonlinear dynamics of solitons with no reflection at the vertex. We also showed that an infinite number of constants of motion are available for NLSE on networks, namely the mapping of Zakharov-Shabat (ZS)'s scheme [24] to networks was achieved.

The extension of the scenario to the discrete NLSE (DNLSE) is far from being obvious. The standard DNLSE is not integrable and the integrable variant of the continuum nonlinear Schrödinger equation is the one proposed by Ablowitz and Ladik (AL) [23, 26–28]. AL equation is the appropriate choice for the zeroth order approximation in studying the soliton dynamics perturbatively in physically motivated models, such as an array of coupled optical waveguides [29] and proton dynamics in hydrogen-bonded chains [30, 31]. The dynamics of intrinsic localized modes in nonlinear lattices can be approximately described by AL equation [32]. Exciton sys-

tems with exchange and dipole-dipole interactions also reduce to AL equation in some limiting cases[33]. The AL chain is integrable by means of the inverse scattering transform, and, together with the Toda lattice [34], constitutes a paradigm of the completely integrable lattice systems.

AL equation for a field variable ψ on one-dimensional (1-d) chain is given by

$$i\dot{\psi}_n + (\psi_{n+1} + \psi_{n-1}) (1 + \gamma|\psi_n|^2) = 0, \quad (1)$$

where γ is the strength of nonlinear inter-site interaction and n denotes each lattice site on the chain. This equation can be obtained from the canonical equation of motion with use of the non-standard Poisson brackets. Equation (1) has an infinite number of independent constants of motion and is completely integrable [26, 27].

However, there is an ambiguity in generalizing the AL model to networks: how can we define the inter-site interaction at each vertex in order to see the infinite-number of constants of motion in networks? To keep the integrability of AL equation, should any rule hold for the strength of nonlinearity on bonds joining at each vertex? We shall resolve these questions in this paper and show how solitons of AL equation on networks will be mapped to that of AL equation on a 1-d chain. Once this mapping will be found, the integrability properties like the inverse scattering transform, Bäcklund transformation, etc, are automatically guaranteed, and will not be addressed in this paper.

Below we shall show the completely integrable case of the AL equation on networks with strength of nonlinearity different from bond to bond. As a relevant issue, with use of reflectionless propagation of AL soliton through networks, we shall evaluate the transmission probabilities on the outgoing bonds. In Section II, using a primary star graph (PSG) and defining a suitable equation of motion at the vertex, we shall address the norm and energy conservations. In Section III, we shall show a basic idea of the soliton propagation along the branched chain, finding the connection formula at the vertex and the sum rule among the strengths of nonlinearity on the bonds, which guarantee the infinite number of constants of motions and complete integrability of the system under consideration. In Section IV, the cases of generalized star graphs and tree graphs are investigated. Section V is devoted to the investigation of an injection of AL soliton which bifurcates at the vertex and is decomposed into a pair of solitons with each propagating along the outgoing

bonds, and we shall evaluate the transmission probabilities on the outgoing bonds. Summary and discussions are given in Section VI.

II. NORM AND ENERGY CONSERVATIONS ON PRIMARY STAR GRAPH

A. Ablowitz-Ladik(AL) equation on networks

Let us consider an elementary branched chain (see Fig.1), namely, a primary star graph (PSG) consisting of three semi-infinite bonds connected at the vertex O .

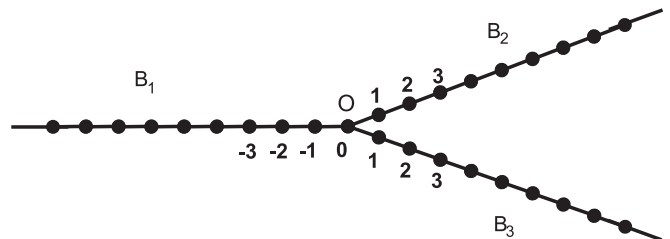


FIG. 1: Primary star graph. 3 semi-infinite chains B_1, B_2 and B_3 connected at a vertex O .

We denote individual lattice sites as (k, n) , where $k = 1, 2$, and 3 mean the bond's number and n corresponds to a lattice site on each bond. For the first bond $k = 1$, n is numbered as $n \in B_1 = \{0, -1, -2, \dots\}$, where $(1, 0)$ means the branching point, i.e., the vertex. For the second ($k = 2$) and third ($k = 3$) bonds, n varies as $n \in B_k = \{1, 2, 3, \dots\}$. $(2, 1)$ and $(3, 1)$ stands for the points nearest to the vertex.

Discrete nonlinear Schrödinger equation (DNLS) *à la* Ablowitz-Ladik (AL) is defined on each bond except for the vicinity of the vertex as

$$i\dot{\psi}_{k,n} + (\psi_{k,n+1} + \psi_{k,n-1}) (1 + \gamma_k |\psi_{k,n}|^2) = 0, \quad (2)$$

where $(k, n) \notin \{(1, 0), (2, 1), (3, 1)\}$. It should be noted that γ_k may be different among bonds. There is an ambiguity about the interaction around the vertex, which is resolved as follows: Let's first introduce Hamiltonian for PSG as

$$H = - \sum_{n=0}^{-\infty} (\psi_{1,n}^* \psi_{1,n+1} + c.c.) - \sum_{k=2}^3 \sum_{n=1}^{+\infty} (\psi_{k,n}^* \psi_{k,n+1} + c.c.), \quad (3)$$

where at the virtual site $(1, 1)$ we assume $\psi_{1,1} = s_2 \psi_{2,1} + s_3 \psi_{3,1}$ with appropriate coefficients s_2 and s_3 . Then

Eq.(2) can be obtained by the equation of motion

$$i\dot{\psi}_{k,n} = \{H, \psi_{k,n}\} \quad (4)$$

at $(k, n) \notin \{(1, 0), (2, 1), (3, 1)\}$, with use of non-standard Poisson brackets

$$\begin{aligned} \{\psi_{k,m}, \psi_{k',n}^*\} &= i(1 + \gamma|\psi_{k,m}|^2)\delta_{kk'}\delta_{mn}, \\ \{\psi_{k,m}, \psi_{k',n}\} &= \{\psi_{k,m}^*, \psi_{k',n}^*\} = 0. \end{aligned} \quad (5)$$

On the same footing as above, the equation of motions in Eq.(4) at $(1, 0)$, $(2, 1)$ and $(3, 1)$ are given, respectively, as

$$i\dot{\psi}_{1,0} + (\psi_{1,-1} + s_2\psi_{2,1} + s_3\psi_{3,1})(1 + \gamma_1|\psi_{1,0}|^2) = 0, \quad (6)$$

$$i\dot{\psi}_{k,1} + (s_k\psi_{1,0} + \psi_{k,2})(1 + \gamma_k|\psi_{k,1}|^2) = 0, \quad k = 2, 3. \quad (7)$$

The solution is assumed to satisfy the following conditions at infinity: $\psi_{1,n} \rightarrow 0$ at $n \rightarrow -\infty$ and $\psi_{k,n} \rightarrow 0$ at $n \rightarrow +\infty$ for $k = 2$ and 3 .

B. Norm and energy conservations

It is known that the norm conservation is one of the most important physical conditions in conservative systems. Since Eqs.(2),(6) and (7) are available from Hamilton's equation of motion with non-standard Poisson brackets, the norm and energy conservations seem obvious. Below, however, we observe them explicitly. Extending the definition in the case of 1-d chain [23], the norm for PSG is given as

$$N = \|\psi\|^2 = \sum_{k=1}^3 \frac{1}{\gamma_k} \sum_{n \in B_k} \ln(1 + \gamma_k|\psi_{k,n}|^2). \quad (8)$$

Its time derivative is given by

$$\frac{d}{dt}N = \sum_{k=1}^3 \sum_{n \in B_k} A_{k,n} \quad (9)$$

with

$$A_{k,n} = \frac{1}{1 + \gamma_k|\psi_{k,n}|^2} \left(\psi_{k,n}^* \dot{\psi}_{k,n} + \dot{\psi}_{k,n}^* \psi_{k,n} \right). \quad (10)$$

For $(k, n) \notin \{(1, 0), (2, 1), (3, 1)\}$ with use of Eq. (2) we have

$$\begin{aligned} A_{k,n} &= \frac{1}{i} (\psi_{k,n} \psi_{k,n+1}^* - \psi_{k,n}^* \psi_{k,n+1}) \\ &\quad - \frac{1}{i} (\psi_{k,n-1} \psi_{k,n}^* - \psi_{k,n-1}^* \psi_{k,n}) \\ &\equiv j_{k,n} - j_{k,n-1}, \end{aligned} \quad (11)$$

where

$$j_{k,n} \equiv \frac{1}{i} (\psi_{k,n} \psi_{k,n+1}^* - \psi_{k,n}^* \psi_{k,n+1}) \quad (12)$$

implies a local current. Firstly one observes

$$\sum_k \sum_n' A_{k,n} = j_{1,0} - j_{2,1} - j_{3,1}, \quad (13)$$

where $\sum_k \sum_n'$ means the summation over all sites on PSG except for the points $(1, 0)$, $(2, 1)$, $(3, 1)$.

Then, for $(k, n) = (1, 0)$, $(2, 1)$, $(3, 1)$, with use of Eqs. (6) and (7) we obtain

$$\begin{aligned} A_{1,0} &= s_2 \frac{1}{i} (\psi_{1,0} \psi_{2,1}^* - \psi_{1,0}^* \psi_{2,1}) \\ &\quad + s_3 \frac{1}{i} (\psi_{1,0} \psi_{3,1}^* - \psi_{1,0}^* \psi_{3,1}) - j_{1,0} \end{aligned} \quad (14)$$

and

$$A_{k,1} = j_{k,1} - s_k \frac{1}{i} (\psi_{1,0} \psi_{k,1}^* - \psi_{1,0}^* \psi_{k,1}) \quad (15)$$

for $k = 2, 3$. Substituting Eqs.(13), (14) and (15) into Eq.(9), we can see $\frac{d}{dt}N = 0$, i.e., the norm conservation. Therefore, for any choice of values s_2 and s_3 the norm conservation turns out to hold well.

On the other hand, the energy for PSG is expressed in a symmetrical form as

$$\begin{aligned} E &= -2\text{Re} \left[\sum_{n=-1}^{-\infty} \psi_{1,n}^* \psi_{1,n+1} + \sum_{k=2}^3 \sum_{n=1}^{+\infty} \psi_{k,n}^* \psi_{k,n+1} \right. \\ &\quad \left. + \psi_{1,0}^* (s_2 \psi_{2,1} + s_3 \psi_{3,1}) \right]. \end{aligned} \quad (16)$$

To show that the energy is conservative, we see its time derivative

$$\begin{aligned} \frac{d}{dt}E &= -2\text{Re} \sum_{n=-1}^{-\infty} \left(\psi_{1,n}^* \dot{\psi}_{1,n+1} + \dot{\psi}_{1,n}^* \psi_{1,n+1} \right) \\ &\quad - 2\text{Re} \sum_{k=2}^3 \sum_{n=1}^{+\infty} \left(\psi_{k,n}^* \dot{\psi}_{k,n+1} + \dot{\psi}_{k,n}^* \psi_{k,n+1} \right) - \\ &\quad - 2\text{Re} \left[\psi_{1,0}^* (s_2 \dot{\psi}_{2,1} + s_3 \dot{\psi}_{3,1}) + \dot{\psi}_{1,0}^* (s_2 \psi_{2,1} + s_3 \psi_{3,1}) \right]. \end{aligned} \quad (17)$$

With use of Eq. (2) we have

$$\begin{aligned} &- \sum_{n=-1}^{-\infty} \left(\psi_{1,n}^* \dot{\psi}_{1,n+1} + \dot{\psi}_{1,n}^* \psi_{1,n+1} \right) \\ &= \frac{1}{i} \sum_{n=-1}^{-\infty} \left[|\psi_{1,n-1}|^2 - |\psi_{1,n+1}|^2 \right] (1 + \gamma_1|\psi_{1,n}|^2) \\ &\quad - \psi_{1,-1}^* \dot{\psi}_{1,0}, \end{aligned} \quad (18)$$

and

$$\begin{aligned}
& - \sum_{n=1}^{\infty} \left(\psi_{k,n}^* \dot{\psi}_{k,n+1} + \dot{\psi}_{k,n}^* \psi_{k,n+1} \right) \\
& = \frac{1}{i} \sum_{n=2}^{\infty} \left[|\psi_{k,n-1}|^2 - |\psi_{k,n+1}|^2 \right] (1 + \gamma_1 |\psi_{k,n}|^2) - \dot{\psi}_{k,1}^* \psi_{k,2}.
\end{aligned} \tag{19}$$

The first terms in the final expressions in Eqs.(18) and (19) are obviously pure-imaginary. Substituting Eqs. (18) and (19) into Eq.(17) and using Eqs.(6) and (7), we find:

$$\begin{aligned}
\frac{d}{dt} E & = -2\text{Re} \left[\psi_{1,0}^* \left(s_2 \dot{\psi}_{2,1} + s_3 \dot{\psi}_{3,1} \right) + \right. \\
& \quad \dot{\psi}_{1,0}^* (s_2 \psi_{2,1} + s_3 \psi_{3,1}) + \psi_{1,-1}^* \dot{\psi}_{1,0} + \\
& \quad \left. + \dot{\psi}_{2,1}^* \psi_{2,2} + \dot{\psi}_{3,1}^* \psi_{3,2} \right] \\
& = 2\text{Re} \left[\frac{1}{i} (1 + \gamma_1 |\psi_{1,0}|^2) (|\psi_{1,-1}|^2 - |s_2 \psi_{2,1} + s_3 \psi_{3,1}|^2) \right. \\
& \quad \left. + \frac{1}{i} \sum_{k=1}^3 (1 + \gamma_k |\psi_{k,1}|^2) (s_k^2 |\psi_{1,0}|^2 - |\psi_{k,2}|^2) \right] = 0.
\end{aligned} \tag{20}$$

The last equality comes from the pure-imaginary nature of the expression in $[\dots]$. Equation (20) is nothing but the energy conservation.

Thus we have proved that the norm and energy are conserved for any choice of values s_2 and s_3 . In general, however, other conservation rules do not hold. In the next sections we shall reveal a special case with appropriate choice of s_2 and s_3 which guarantees an infinite number of conservation laws.

III. COMPLETELY INTEGRABLE CASE

A. Dynamics near branching point and sum rule

Among many possible choices of s_2 and s_3 , there is one special case in which an infinite number of constants of motion can be found and DNLSE on PSG becomes completely integrable. To investigate this case, we shall first add to each bond B_k ($k = 1, 2, 3$) a ghost-bond counterpart B'_k so that $B_k + B'_k$ constitutes an ideal 1-d chain (see Fig. 2). Then we suppose that the soliton solution of AL equation on PSG is given by

$$\psi_{k,n}(t) = \frac{1}{\sqrt{\gamma_k}} q_{k,n}(t), \quad k = 1, 2, 3 \tag{21}$$

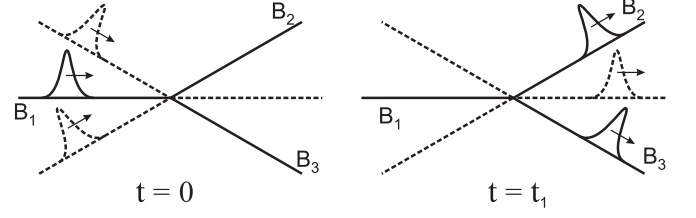


FIG. 2: Real bonds and real solitons (solid lines) and ghost bonds and ghost solitons (broken lines).

where $q_{k,n}(t)$ are soliton solutions of DNLSE with unit nonlinearity on the ideal 1-d chain ([23, 26, 27]):

$$i\dot{q}_n + (q_{n+1} + q_{n-1})(1 + |q_n|^2) = 0, \tag{22}$$

with n being integers in $(-\infty, +\infty)$. The solutions of Eq.(22) may be different among three fictitious chains $B_k + B'_k$ ($k = 1, 2, 3$).

Comparing Eqs. (6), (7) and (22), one can find at the vertex the following two equalities:

$$\frac{1}{\sqrt{\gamma_1}} q_{1,1}(t) = \frac{s_2}{\sqrt{\gamma_2}} q_{2,1}(t) + \frac{s_3}{\sqrt{\gamma_3}} q_{3,1}(t), \tag{23}$$

$$\frac{1}{\sqrt{\gamma_k}} q_{k,0}(t) = \frac{s_k}{\sqrt{\gamma_1}} q_{1,0}(t), \quad k = 2, 3. \tag{24}$$

Noting the spatio-temporal behavior of soliton solutions and to guarantee the equality in Eq.(24), $q_{k,n}(t) = s_k \sqrt{\frac{\gamma_k}{\gamma_1}} q_{1,n}(t)$ with $k = 2, 3$ should be satisfied for any time t and for any integer n , from which we obtain

$$s_k \sqrt{\frac{\gamma_k}{\gamma_1}} = 1 \quad \text{or} \quad s_k = \sqrt{\frac{\gamma_1}{\gamma_k}} \quad (k = 2, 3) \tag{25}$$

and

$$q_{k,n}(t) \equiv q_n(t), \tag{26}$$

namely, the solution $q_{k,n}(t)$ should be bond-independent. With use of Eqs.(25) and (26) in Eq. (23) we have the sum rule among nonlinearity coefficients γ_1 , γ_2 and γ_3 :

$$\frac{1}{\gamma_1} = \frac{1}{\gamma_2} + \frac{1}{\gamma_3} \tag{27}$$

Equations (25), (26) and (27) are the necessary and sufficient conditions to see Eqs.(23) and (24). Thus, under the sum rule for nonlinearity coefficients in Eq.(27), the solution on PSG is given by a common (bond-independent) soliton solution of Eq.(22) multiplied by square root of the inverse nonlinearity coefficient. For example, the soliton incoming through the bond B_1 is

expected to smoothly bifurcate at the vertex and propagate through the bonds B_2 and B_3 , as we shall see in Fig. 4. In the case that γ_1, γ_2 and γ_3 break the sum rule, we shall see a completely different nonlinear dynamics of solitons such as their reflection and emergence of radiation at the vertex, as will be shown in Fig.6. The initial value problem for such a case is outside the scope of the present work.

We also note that the parameters s_2 and s_3 would correspond to $\frac{\alpha_2}{\alpha_1}$ and $\frac{\alpha_3}{\alpha_1}$, respectively, in the preceding work [25], although the derivations of the connection formula at the vertex are quite different between the continuum and discrete systems. In fact, s_2 and s_3 are introduced to define the inter-site interaction at the vertex and are not obtained from the norm and energy conservations, in contrast to the case of networks consisting of continuum segments[25].

B. An Infinite number of constants of motion

It is well known that Ablowitz-Ladik (AL) equation on the 1-d chain has an infinite number of constants of motion. Now we shall proceed to obtain an infinite number of constants of motion for general solutions of AL equation on PSG. First of all, it should be noted that the solution on PSG can now be written as

$$\psi_k(t) = \frac{1}{\sqrt{\gamma_k}} \{q_n(t) | n \in B_k\}, \quad k = 1, 2, 3, \quad (28)$$

where $q(t)$ stands for a general solution of AL equation (22) and is restricted to each bonds B_k ($k = 1, 2, 3$).

While we already proved the conservation of energy, we can generalize it to the general case: Without taking the complex conjugate, Eq. (3) can be explicitly written as

$$Z = - \sum_{n=-1}^{-\infty} \psi_{1,n}^* \psi_{1,n+1} - \sum_{k=2}^3 \sum_{n=1}^{+\infty} \psi_{k,n}^* \psi_{k,n+1} - \psi_{1,0}^* (s_2 \psi_{2,1} + s_3 \psi_{3,1}). \quad (29)$$

Substituting Eq.(28) into Eq.(29), Z is rewritten as

$$Z = - \frac{1}{\gamma_1} \sum_{n=0}^{-\infty} q_n^* q_{n+1} - \sum_{k=2}^3 \frac{1}{\gamma_k} \sum_{n=1}^{+\infty} q_n^* q_{n+1} + \frac{1}{\gamma_1} q_0^* q_1 - \sum_{k=2}^3 \frac{s_k}{\sqrt{\gamma_1 \gamma_k}} q_0^* q_1. \quad (30)$$

Using the value s_k in Eq.(25) and the sum rule in Eq.(27), Eq.(30) reduces to the constant for the ideal 1-d chain

[26, 27]:

$$Z = - \frac{1}{\gamma_1} \sum_{n=-\infty}^{+\infty} q_n^* q_{n+1}. \quad (31)$$

Therefore Z in Eq.(29) is a constant of motion, and its real and imaginary parts imply the energy and current, respectively.

For other higher-order conservation rules, we can write them as

$$\begin{aligned} \frac{1}{\gamma_1} C_m = & \frac{1}{\gamma_1} \sum_{n=0}^{-\infty} f_m^{(n)} (\{q_n | n \in B_1\}) \\ & + \sum_{k=2}^3 \frac{1}{\gamma_k} \sum_{n=1}^{+\infty} f_m^{(n)} (\{q_n | n \in B_k\}), \end{aligned} \quad (32)$$

with f_m defined as expansion coefficients of the expression (see Ablowitz & Ladik [27])

$$\log(g_n^{(0)} + g_n^{(1)} z^2 + g_n^{(2)} z^4 + \dots) = f_1^{(n)} z^2 + f_2^{(n)} z^4 + \dots, \quad (33)$$

where $(g_n^{(m)})$ are given by

$$\begin{aligned} g_n^{(0)} &= 1, \quad g_n^{(1)} = R_{n-1} Q_{n-2}, \\ g_n^{(m)} &= \frac{R_{n-1}}{R_{n-2}} g_{n-1}^{(m-1)} - \sum_{l=1}^{m-1} g_{n-1}^{(m-l)} g_n^{(l)}, \quad m = 2, 3, 4, \dots, \end{aligned} \quad (34)$$

$$R_n = q_{n+2}^*, \quad Q_n = -q_{n+2}. \quad (35)$$

The relations (34) and (35) are obtained by solving Eq. (4.15) in [27], i.e.,

$$g_{n+1}(g_{n+2} - 1) - z^2 \frac{R_{n+1}}{R_n} (g_{n+1} - 1) = z^2 R_{n+1} Q_n, \quad (36)$$

recursively with use of the expansion

$$g_n = g_n^{(0)} + g_n^{(1)} z^2 + g_n^{(2)} z^4 + \dots. \quad (37)$$

The right-hand side of Eq. (32) includes some undefined field variables in the ghost bond regions which must be defined as

$$\begin{aligned} \psi_{1,n} &= \sqrt{\frac{\gamma_1}{\gamma_2}} \psi_{2,n} + \sqrt{\frac{\gamma_1}{\gamma_3}} \psi_{3,n} \quad \text{with } n \geq 1, \\ \psi_{k,n} &= \sqrt{\frac{\gamma_1}{\gamma_k}} \psi_{k,n}, \quad k = 2, 3 \quad \text{with } n \leq 0. \end{aligned} \quad (38)$$

The conservation laws in Eq. (32) follows from the nature of solutions (28) and the sum rule for nonlinearity coefficients (27).

For $m = 1$ we obtain current and energy conservation laws. At $m \geq 2$ we obtain higher order conservation laws. Some of higher-order constants of motion are as follows:

$$\frac{1}{\gamma_1} C_2 = - \sum_{k=1}^3 \sum_{n \in B_k} \left(\psi_{k,n+1}^* \psi_{k,n-1} (1 + \gamma_k |\psi_{k,n}|^2) + \frac{\gamma_k}{2} \psi_{k,n}^2 (\psi_{k,n+1}^*)^2 \right), \quad (39)$$

$$\begin{aligned} \frac{1}{\gamma_1} C_3 = & - \sum_{k=1}^3 \sum_{n \in B_k} \left[(\psi_{k,n+2}^* \psi_{k,n-1} (1 + \gamma_k |\psi_{k,n+1}|^2) + \right. \\ & + \gamma_k \psi_{k,n}^* \psi_{k,n+1}^* \psi_{k,n-1}^2 \\ & + (\psi_{k,n+1}^*)^2 \psi_{k,n} \psi_{k,n-1}) (1 + \gamma_k |\psi_{k,n}|^2) \\ & \left. + \frac{\gamma_k^2}{3} \psi_{k,n+1}^* \psi_{k,n} \right], \quad (40) \end{aligned}$$

where field variables at lattice sites of the ghost bonds are defined in Eq. (38).

IV. GENERALIZED STAR AND TREE GRAPHS

Now we proceed to explore soliton solutions of DNLSE on other types of graphs and explore the sum rule and conservation rules for solitons to propagate through these graphs.

The above treatment on PSG is also true for more general star graphs consisting of N semi-infinite bonds connected at a single vertex. In such cases, the initial soliton at an incoming bond B_1 splits into $N - 1$ solitons in the remaining bonds, and the extended version of Eq. (27) is

$$\frac{1}{\gamma_1} = \sum_{j=2}^N \frac{1}{\gamma_j}. \quad (41)$$

The solution is given by the equations

$$\psi_{k,n}(t) = \frac{1}{\sqrt{\gamma_k}} q_n(t), \quad (42)$$

where $n = 0, -1, -2, \dots$ for the first bond ($k = 1$) and $n = 1, 2, 3, \dots$ for other bonds ($2 \leq k \leq N$). $q_n(t)$ is a soliton solution of Eq. (22). Conservation laws for this graph can be obtained analogously as in the case of PSG.

Another example of the graph for which the soliton solution of DNLSE can be obtained analytically is a tree graph in Fig. 3. Now we shall provide a soliton solution in this case. We denote bonds of graph as

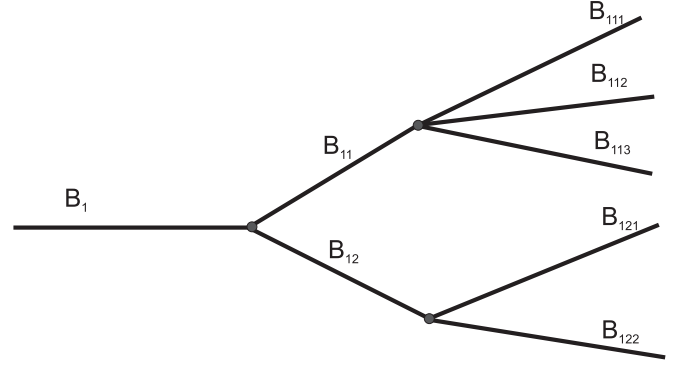


FIG. 3: Tree graph. $B_1 \sim (-\infty, 0)$, $B_{11}, B_{12} \sim (0, L)$, and $B_{1ij} \sim (0, +\infty)$ with $i, j = 1, 2, \dots$.

$B_\Lambda = B_{1ij\dots m}$ and number the lattice sites on this bonds as $1, 2, 3, \dots, N_\Lambda$. On each branching point we assume the following conditions are hold

$$\frac{1}{\gamma_\Lambda} = \sum_m \frac{1}{\gamma_{\Lambda m}}. \quad (43)$$

The solution is given by

$$\psi_{\Lambda,n}(t) = \frac{1}{\sqrt{\gamma_k}} q_{n+s_\Lambda}(t), \quad n \in B_\Lambda. \quad (44)$$

Here s_Λ number of lattice sites that soliton pass through from B_1 to B_Λ . For the tree graph it is defined as

$$\begin{aligned} s_1 &= s_{1i} = n_0, \quad s_{1ij} = n_0 + N_{1i}, \\ s_\Lambda &\equiv s_{1ij\dots lm} = n_0 + N_{1i} + \dots + N_{1ij\dots l}. \end{aligned} \quad (45)$$

Below, applying the induction method we give a proof of conservation laws for soliton solutions of AL on tree graph. Let us denote the tree graph as G and assume the conservation laws to hold in G : $\sum_{B_\Lambda \in G} \sum_{n \in B_\Lambda} f_n^{(k)}(q_{n+s_\Lambda}(t)) = \text{const.}$ Then we construct an enlarged tree graph in the following way: First we choose the arbitrary point N_Φ in the one of right-most semi-infinite chain B_Φ as a new branching point. Cut off semi-infinite part of this bond at the point N_Φ and attach M semi-infinite bonds to this point. Namely the bond B_Φ is now replaced by a finite bond \tilde{B}_Φ connected with M semi-infinite bonds $B_{\Phi m} = \{1, 2, \dots, N_{\Phi m}\}$, with $m = 1, 2, \dots, M$. For the enlarged tree graph, constants

of motion are given by

$$\begin{aligned}
& \sum_{B_\Lambda \in G-B_\Phi} \gamma_\Lambda^{-1} \sum_{n \in B_\Lambda} f_n^{(k)}(q_{n+s_\Lambda}(t)) + \gamma_\Phi^{-1} \sum_{n \in \bar{B}_\Phi} f_n^{(k)}(q_{n+s_\Phi}(t)) \\
& + \sum_{m=1}^M \gamma_{\Phi m}^{-1} \sum_{n \in B_{\Phi m}} f_n^{(k)}(q_{n+s_\Phi+N_\Phi}(t)) \\
& = \sum_{B_\Lambda \in G-B_\Phi} \gamma_\Lambda^{-1} \sum_{n \in B_\Lambda} f_n^{(k)}(q_{n+s_\Lambda}(t)) + \gamma_\Phi^{-1} \sum_{n=1}^{N_\Phi} f_n^{(k)}(q_{n+s_\Phi}(t)) \\
& + \sum_{m=1}^M \gamma_{\Phi m}^{-1} \sum_{n=1+N_\Phi}^{+\infty} f_n^{(k)}(q_{n+s_\Phi+N_\Phi}(t)) \\
& = - \left(\gamma_\Phi^{-1} - \sum_{m=1}^M \gamma_{\Phi m}^{-1} \right) \sum_{n=1+N_\Phi}^{+\infty} f_n^{(k)}(q_{n+s_\Phi+N_\Phi}(t)) + \text{const.}
\end{aligned} \tag{46}$$

It is clear that the final expression becomes constant under the sum rule (43). Thus, starting from PSG in Fig. 1 and repeating the above procedure, we can get the conservation rule for all tree graphs.

V. TRANSMISSION PROBABILITIES AGAINST INJECTION OF A SINGLE SOLITON

A relevant issue of the above discoveries is the transmission probability against injection of a single soliton. Here we calculate transmission probabilities for a single soliton which is incoming through a semi-infinite bond B_1 and outgoing through the other semi-infinite bonds $\{B_l | l \neq 1\}$.

A single (bright) soliton on a graph, which takes the general form as in Eqs. (28), (42) and (44), is described with use of AL soliton with $\gamma = 1$ [26]: $\psi_{l,n}(t)$ lying on individual bonds B_l is given by

$$\begin{aligned}
\psi_{l,n}(t) &= \gamma_l^{-1/2} \sinh \beta \text{sech}[\beta(n - n_0 - vt)] \\
&\quad \times e^{-i(\omega t + \alpha n + \phi_0)}, \\
n &\in B_l, \quad l = 1, 2, 3, \dots, N,
\end{aligned} \tag{47}$$

where $\omega = -2\cosh\beta \cos\alpha$, $v = -(2/\beta)\sinh\beta \sin\alpha$, $-\pi \leq \alpha \leq \pi$, $0 < \beta < \infty$, $0 \leq \phi_0 < 2\pi$ and n_0 are bond-independent parameters characterizing frequency, velocity, wave number, inverse width of the soliton, initial phase and initial center of mass, respectively. Equation (47) indicates that a narrow soliton travels faster than wider ones with the same α .

It should be noted that parameter values are common to each bond, except for $\{\gamma_l\}$. Choosing the simplest

network PSG in Fig.1, we shall give conservative quantities for the solution in Eq. (47) under the sum rule in Eq.(27). First of all, the norm in Eq.(8) turned out to be reduced to the one for the 1-d chain with the nonlinearity constant γ_1 and thereby is given by

$$N = 2\beta/\gamma_1. \tag{48}$$

Equation (48) indicates that a narrow soliton has a larger norm than wider ones. As for the energy (E) and current (J), it is convenient to evaluate the combined quantity Z in Eq. (29) with use of s_2 and s_3 given by Eq. (25). In fact we have

$$E = -2\text{Re}(Z), \quad J = 2\text{Im}(Z). \tag{49}$$

Substituting Eq. (47) into Eq. (29) and using the sum rule in Eq. (27), one obtains

$$Z = \frac{2}{\gamma_1} e^{-i\alpha} \sinh \beta \tag{50}$$

and

$$E = -\frac{4}{\gamma_1} \cos \alpha \sinh \beta, \quad J = -\frac{4}{\gamma_1} \sin \alpha \sinh \beta. \tag{51}$$

As is seen from Eq. (47), the center of mass of the soliton (CMS) on each bond B_l is located at $n = n_0$ at $t = 0$. However, lattice points on the individual semi-infinite bonds are defined on the limited interval. In particular, on outgoing bonds $\{B_l | l \neq 1\}$, their lattice points n are defined in the interval $(1, +\infty)$. If $n_0 < 0$, therefore, CMS on $\{B_l | l \neq 1\}$ is initially located outside of the real bonds. In such cases we call the soliton as a "ghost soliton". When CMS belongs to a real bond we use a term "real soliton". In Fig. 2 which corresponds to PSG in Fig.1, ghost solitons are plotted by broken curve while real ones by solid line. The soliton dynamics here is governed by a single characteristic time $\tau \equiv \frac{-n_0}{v}$. While for $0 \leq t \leq \tau$ the soliton at B_1 is a real one and those at B_2 and B_3 are ghosts, for $\tau \leq t$ the soliton at B_1 is a ghost and those at B_2 and B_3 are real. At $t = 0$ with $-n_0 \gg 1$, the soliton lying on the bond B_1 is exclusively responsible for the norm N . On the other hand, at $t \gg 1$, the solitons running through the bonds B_2 and B_3 are exclusively responsible for the norm. Therefore we can naturally define transmission probabilities at $t \rightarrow +\infty$.

In general networks, transmission probability for an arbitrary semi-infinite bond $B_l (l \neq 1)$ at discrete time \hat{t}

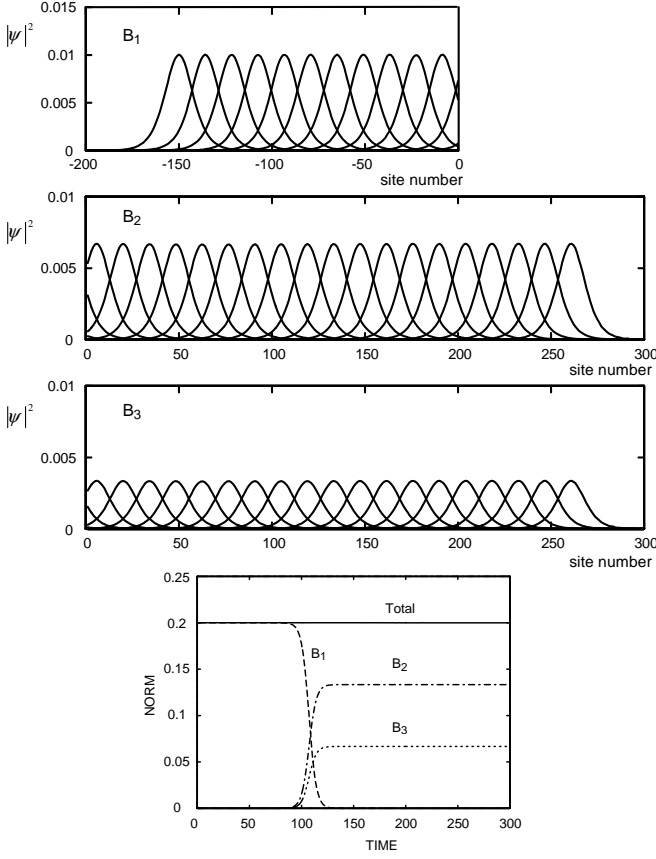


FIG. 4: Numerical result for time evolution of a soliton propagation through a vertex in PSG. Strength of nonlinearity at each bond are $\gamma_1 = 1$, $\gamma_2 = 1.5$, $\gamma_3 = 3$ satisfying the sum rule in Eq.(27). Space distribution of wave function probability is depicted in every time interval $T = 10.0$ with time used commonly in branches 2 and 3. Abscissa represents discrete lattice coordinates defined in Fig.1. Initial profile is Ablowitz-Ladik soliton in Eq.(47) at $t = 0$ with parameters $\beta = 0.1, \alpha = 5\pi/4$. Time difference in numerical iteration is $\Delta t = 0.01$ Bottom panel shows the time dependence of partial norms at each of 3 branches.

that makes $v\hat{t}$ integers are defined as

$$\begin{aligned}
 T_l &= \frac{1}{N\gamma_l} \sum_{n=1}^{+\infty} \ln(1 + \gamma_l |\psi_{l,n}|^2) = \\
 &= \frac{1}{N\gamma_l} \sum_{n=1}^{+\infty} \ln(1 + \sinh^2 \beta \operatorname{sech}^2(\beta(n - n_0 - v\hat{t}))) \\
 &= \frac{\gamma_1}{N\gamma_l} \sum_{n'=1-n_0-v\hat{t}}^{+\infty} \frac{1}{\gamma_1} \ln(1 + \sinh^2 \beta \operatorname{sech}^2(\beta n')).
 \end{aligned} \tag{52}$$

At $v\hat{t} \rightarrow +\infty$, $\sum_{n'=1-n_0-v\hat{t}}^{+\infty}$ on the last line in Eq.(52) tends to $\sum_{n'=-\infty}^{+\infty}$ and this summation gives N , i.e., the normalization of the soliton in the ideal 1-d chain with

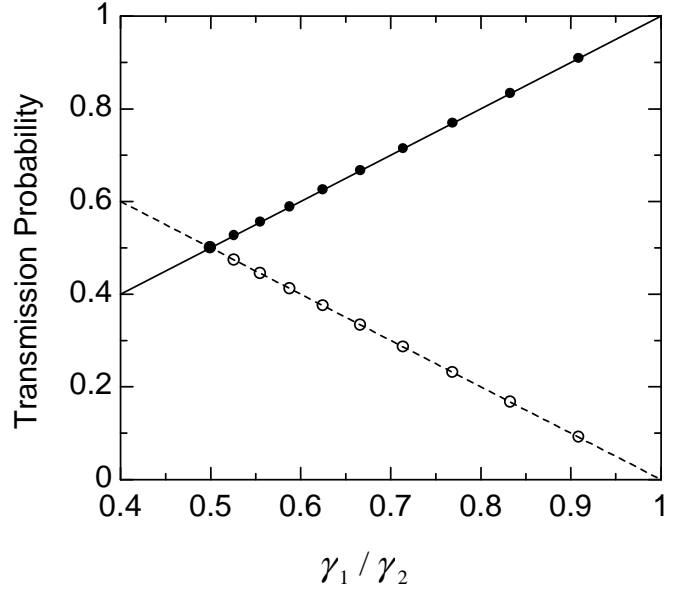


FIG. 5: Transmission probabilities (TP) as a function of $\frac{\gamma_1}{\gamma_2}$ in PSG. Symbols and lines denote numerical and theoretical results, respectively. A solid line with \bullet and a broken line with \circ correspond to T_2 and T_3 , respectively.

the nonlinearity coefficient γ_1 . Therefore

$$T_l = \frac{\gamma_1}{\gamma_l} \tag{53}$$

Under the sum rules as in Eqs. (27), (41) and (43) we have the unitarity condition

$$\sum_{l=2}^N T_l = 1, \tag{54}$$

where the summation is taken over the semi-infinite bonds except for B_1 . The result in Eq.(53) means that the transmission probability is inversely proportional to the strength of nonlinearity in outgoing semi-infinite bonds.

We have checked this result using a numerical simulation of the discrete nonlinear Schrödinger equation (DNLS) on PSG in Fig.1: We numerically iterated Eqs.(2), (6) and (7) with use of Eq.(25) and chose the initial profile in Eq.(47) with γ_1 and $n_0 = -150$ as an incoming soliton. Figure 4 shows the result in the case that the sum rule in Eq.(27) is satisfied: the soliton starting at lattice point $n = -150$ in the branch 1 enters the vertex at $n = 0$ and is smoothly split into a pair of smaller solitons in the branches 2 and 3 with no reflection at the vertex. The velocity and width of the soliton have the definite value common to all bonds, and the squared peak

value of the soliton is proportional to γ_k , which are consistent with the result in Eq.(47). Bottom panel in Fig.4 shows the time dependence of partial norms at each of 3 branches. With increasing time, the partial norms at branches 2 and 3 converge to the transmission probabilities in Eq.(53).

In Fig.5 transmission probabilities T_2 and T_3 are plotted as a function of $\frac{\gamma_1}{\gamma_2}$ in the wider range of γ_1 and γ_1 in the case satisfying the sum rule in Eq.(27). We can confirm the linear law predicted in Eq.(53).

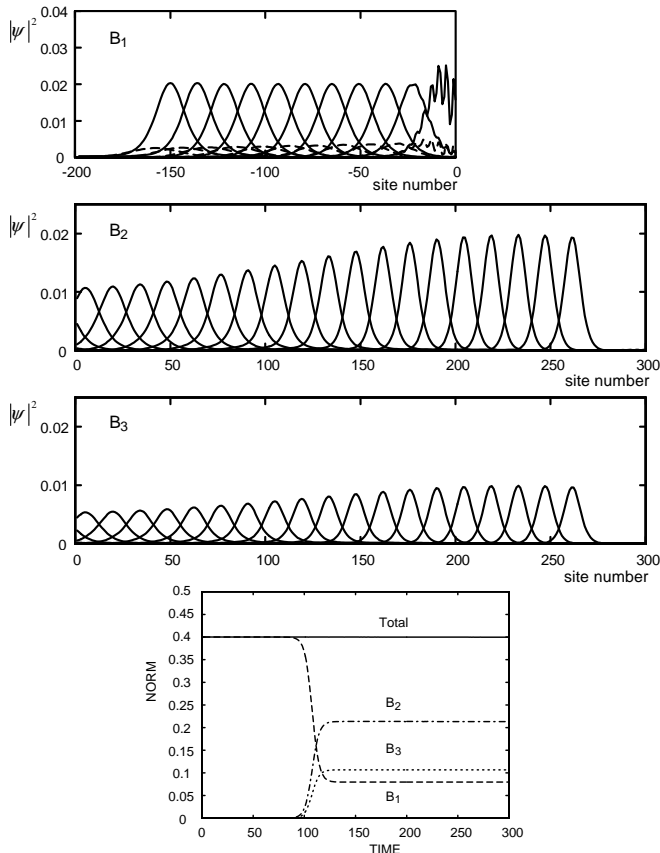


FIG. 6: Numerical result for time evolution of a soliton propagation through a vertex in the case of $\gamma_1 = 0.5, \gamma_2 = 1.5, \gamma_3 = 3$, which breaks the sum rule. Initial profile and parameter values are the same as in Fig.4. In top panel, broken curves indicate a propagation of the reflected soliton. Bottom panel shows the time dependence of partial norms at each of 3 branches.

Figure 6 shows the result in the case that the sum rule is broken: $\frac{\gamma_1}{\gamma_2} + \frac{\gamma_1}{\gamma_3} \neq 1$. In this case the soliton starting at lattice point $n = -150$ in the branch 1 enters the vertex at $n = 0$, but is accompanied with both reflection and emergence of radiation at the vertex. It is very inter-

esting that the velocity of the self-organized soliton have the definite value common to all bonds. In particular, the reflected soliton at the branch 1 has the same magnitude of velocity as that of the incident soliton. With increasing time, the partial norms at branches 1, 2 and 3 would converge to the reflection (on B_1) and transmission probabilities (on B_2 and B_3). For some other choice of γ_1, γ_2 and γ_3 that breaks the sum rule (which is not shown here), the asymptotically ($t \gg 1$)-equal velocity of solitons running on all three semi-infinite bonds can also be observed and provides an open question to be resolved in due course.

VI. SUMMARY AND DISCUSSIONS

We have derived conditions under which Ablowitz-Ladik (AL) type discrete nonlinear Schrödinger equation (DNLSE) on simple networks is mapped to the original one on the ideal 1-d chain and becomes completely integrable. Here the strength of cubic nonlinearity is different from bond to bond, and networks are assumed to have at least two semi-infinite bonds with one of them used as an incoming bond. Our findings are: (1) the solution on each bond is a part of the universal (bond-independent) soliton solution of the completely-integrable DNLSE on the 1-d chain, but is multiplied by the inverse of square root of bond-dependent nonlinearity; (2) the inverse nonlinearity at an incoming bond should be equal to the sum of inverse nonlinearities at the remaining outgoing bonds; (3) with use of the above two findings, there exist an infinite number of constants of motion. The parameters s_2 and s_3 , which played an essential role in deriving the connection formula, are introduced to define the inter-site interaction at the vertex and are not obtained from the norm and energy conservations, in marked contrast to the case of networks consisting of continuum segments[25]. The argument on a branched chain or a primary star graph (PSG) is generalized to general star graphs and tree graphs by using the induction method. As a practical issue, with use of AL soliton injected through the incoming bond, we obtain transmission probabilities inversely proportional to the strength of nonlinearity on the outgoing bonds.

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